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ON RECYCLING AND TECHNOLOGICAL EXTERNALITIES

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On Recycling and Technological Externalities*

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Abstract

This article shows that, apart from the environment-related externalities linked to waste management and recycling, which are reported in the previous literature, the technological aspect of recycling is an additional source of externalities. The main focus of the paper is the impact that the presence of recycling has on the technological profile of the economy and the implications of this change for the dynamic paths of the key economic variables. We show that disregarding (or not completely internalizing) the technological effect of recycling can result in dramatic consequences regarding the dynamic evolution of production and consumption decisions. A generalization of the traditional concept of production function is proposed: the Production and Recycling Function (PRF). This function provides an integrated view of conventional production and recycling and represents the production set of the economy when a recycling technology is available.

JEL classification: Q20, Q30.

Keywords: recycling, natural resources, production set, externalities

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1 Introduction

The increasing amount of waste has become an urgent problem in many countries (see, for example, Quadrio-Curzio et. al. (1994), Beede and Bloom (1995), Porter (2002) or Fullerton and Kinnaman (2002). The most traditional way to handle waste is landfill disposal[1] but the increasing cost of adequate landfill space, as well as other environmental reasons, has triggered the search for alternative treatment methods and, specifically, for ways to re-use and recycle waste (see Anderson, 1987; Dinan, 1993; Highfill et. al., 1994; Huhtala, 1997, 1999).

Waste management, and specifically the possibility of recycling residuals, is subject to important economic externalities. On the one hand, Weinstein and Zeckhauser (1974), Schulze (1974) Lusky (1975a,b), Hoel (1978), Dinan (1993), Huhtala (1994) remark that the use of recycled materials enables scarce resources to be saved. On the other hand, as discussed in Smith (1972), Lund (1990), Sigman (1995), Ready and Ready (1995), Highfill and McAsey (1997), Huhtala (1997, 1999), recycling is a waste treatment technique which is more environmentally friendly than other alternatives, such as landfilling or incineration, and allows landfill space to be saved. Insofar as recycling generates some social (environmental) benefits that do not accrue to any specific individual, there is some externality which leads to the amount of recycling in the competitive market not being efficient. As a consequence, some kind of policy instrument is typically needed to restore efficiency (see Dinan(1993), Sigman (1995), Fullerton and Kinnaman (1995, 1996), Palmer and Walls (1997) Shinkuma (2003)).

This article stresses a further aspect of recycling: it can be regarded as a particular type of productive technology having an important effect on the production set of the economy. Insofar as the markets related to waste management and recycled products are not perfect, the technological impact of recycling is likely not to be fully internalized. As a consequence, we obtain a further source of economic externalities specifically linked to the technological nature of recycling, that add to the conventional environmental externalities reported in the literature. To distinguish them from the latter, we label the former as "technological externalities".

Recycling becomes more technically difficult to perform (and hence less effective in practice) if waste contains a complicated composition of different materials instead of being made of a simple material. As a matter of fact, separation and sorting of waste is one of the main tasks involved in recycling activities,

[1]Some recent papers have addressed landfill management from an economic point of view. See, for example, Ready and Ready (1995), Gaudet et al. (2001), André and Cerdá (2001b, 2003), Ley et.al (2002).

and so waste mixing becomes a key practical difficulty for the profitability and viability of recycling. Henceforth, in order to determine the optimal use of a resource, it is crucial to take into account its recycling ability and its effect on the recycling possibilities of other materials. This idea is captured in the theoretical model presented below by specifying a recycling technology that depends on the material composition of waste.

We can argue that the problem of waste management (and hence the use of recycling) does not begin with the waste flow emanating from consumption, but it begins at a previous stage, when production decisions are made. This observation has already been implicitly or explicitly taken into account by several authors (see, for example, Dinan (1993), Pearce and Brisson (1994) Bruvoll (1998)). We can conclude that the possibility of recycling not only changes the optimal treatment of waste, but also the optimal production decisions. On the one hand, recycling increases the effective availability of resources. On the other hand, it introduces a new channel for technological interaction among different productive resources.

As shown in Beckman (1974, 1975) and Hartwick (1978a, 1978b, 1990), productive processes usually depend on several natural resources in such a way that it is possible to choose among different resource combinations. Hartwick (1978a) obtains some results regarding substitution among nonrenewable resources. André and Cerdá (2001a) study the optimal substitution among different natural resources whether they are renewable or nonrenewable. The present article uses a dynamic model to analyze the optimal production and substitution among different natural (renewable and/or nonrenewable) resources when a recycling technology is present. Interestingly, the presence of recycling involves some further channels for technical interaction among different resources, resulting, not only in an increment in the effective availability of the recyclable resources (which can be seen as a pure *scale effect*), but also in a dramatic change concerning the dynamic properties of the optimal solution. As a consequence, disregarding the possibility of recycling may involve totally different dynamic patterns for the consumption and production decisions (which we can label as *technological effect*).

The technological nature of recycling has been pointed out by Huhtala (1999) and Di Vita (2001) but, to the best of our knowledge, there is not any paper that has explicitly modeled the implications of the material composition of output (and hence, of waste) for the recycling possibilities and the resulting interactions between conventional production and recycling.

The remainder has the following structure. In section 2 the theoretical model is presented. We are interested in showing that, apart from the environment-related externalities linked to recycling, there are some technological externalities that could arise even if the first ones were not present. We deliberately construct a very simple model in order to highlight the role of the technological nature of recycling and its interaction with the production decisions. For that purpose, we do not model other (environmental) externalities related to waste management and recycling (that have already been studied in the literature, as discussed above). Furthermore, we concentrate on the production decisions involving natural resources (materials and energy) that are the inputs which are subject to the possibility of recycling. So, we take as given the rest of the relevant inputs such as labor and capital.

Section 3, for comparison purposes, shows the main results regarding optimal output and relative use of resources in the absence of recycling. These results are compared, in section 4, with the parallel ones when recycling is available, so that the effect of such a technology is pointed out. We show that, even if environmental concern is not explicitly taken into account, the simple fact that a recycling technology is available is a source of externalities, so that the market outcome is not likely to be efficient. Specifically, there are at least two possible market failures, one stemming from the recyclability of some resource itself, and the other coming from the possible technical interactions among resources because of the mixing of residuals. As a consequence, in general at least two policy instruments are needed. We mainly focus on the different dynamic consequences of the introduction of recycling. We show that, in the solution, the optimal output path follows a version of the Keynes-Ramsey rule where the marginal productivity of capital is replaced by the "marginal productivity of natural capital", that results from a weighted sum of the marginal growth of both resources according to their weight on the aggregate technology of the economy. The substitution between two resources depends on the difference between their marginal growth and the flexibility of the technology. Nonrenewable resources are always used in a constant proportion that depends on their relative weight in production and their relative scarcity. The results involving renewable and nonrenewable resources provide some insight concerning the contribution of recyclable resources and renewable resources for the sake of sustainability. Mainly: if production depends on renewable and recyclable resources, the recoverability of the latter increases its effective availability and permits a more intense use in the short term. In the long term, however, the possibility of obtaining sustainable paths crucially depends on the extent to which production rests more and more intensively on renewable resources, so that, from the viewpoint of sustainability, the presence of a recycling technology is not enough by itself to compensate for the exhaustibility of non-renewable resources.

To illustrate the economic dynamic implications of recycling, section 5 offers an illustrative example, with specific production and recycling functions, in which we show that the technological impact of recycling can result in totally different time patterns for consumption and production decisions. Section 6 suggests a way to approach both the conventional technology and recycling technology in an integrated framework. For that purpose, we introduce the new concept of Production and Recycling Function (PRF), which is a generalization of the traditional production function, representing the new production set when recycling is taken into account. This new concepts helps to complete and interpret the results obtained in the previous sections. The main conclusions and some further research lines are given in section 7 and all the mathematical results of the article are proved in an appendix in section 8.

2 Model and Assumptions

Assume an economy with a single consumption good, whose quantity is denoted by $Y \geq 0$, obtained from two natural resources used as inputs in quantities $X_1 \geq 0$ and $X_2 \geq 0$, in agreement with the production function $Y = F(X_1, X_2)$, which is assumed to be of class $C^{(2)}$, homogeneous of degree 1, and verifying F_1, $F_2 > 0$, F_{11}, $F_{22} < 0$, $F_{11}F_{22} - F_{12}^2 > 0$, where F_i denotes the partial derivative $\frac{\partial F}{\partial X_i}$ and F_{ij} denotes $\frac{\partial^2 F}{\partial X_i \partial X_j}$. Let us define the relative use of resources as the ratio $x = \frac{X_1}{X_2}$. In order to focus our attention on natural resources, we take as exogenously given the quantities of any other inputs, such as capital and labor. Furthermore, a model with two resources is rich enough to address the main questions raised in this paper. The solution provides simple and economically meaningful results that can be useful to manage any arbitrary number of resources.

The whole output Y is consumed by a single consumer in the economy, whose preferences are represented by the utility function $U(Y)$, which we assume is of class $C^{(2)}$ and verifies $U' > 0$, $U'' < 0$. X_i $(i = 1, 2)$ is extracted from the available natural stock of resource i, denoted by S_i. This stock grows in accordance with its natural growth function $g_i(S_i)$ which is concave, of class $C^{(2)}$ and verifies $g_i(0) = 0$. As noted for example in Smith (1968), the nonrenewable case can be seen as a particular one with $g_i(S_i) = 0 \; \forall \, S_i$.

After consumption, some waste is generated. This waste can be recycled, depending on its composition, to recover a certain amount R^1 of resource 1. Assume, as an example, that the model is meant to represent the production of boxes, so the final output Y is measured in "number of boxes", which are produced using both paper (X_1) and plastic (X_2). A recycling technology is available for paper, but not for plastic. Then, the amount of paper that can be recovered by means of recycling should not depend on the number of boxes (Y), but on the amount of paper (X_1) and plastic (X_2) contained in the boxes. This observation is central for the model and the results displayed in the paper and we represent it by assuming that the output of the recycling process is given by $R^1 = R(X_1, X_2)$, where R is a function of class $C^{(2)}$, homogeneous of degree 1, and satisfying the assumptions $0 \leq R(X_1, X_2) < X_1$ (it is not possible to recover a quantity of resource 1 equal to or larger than that employed in production) and $0 \leq R_1 < 1$ (the recovered quantity increases with X_1 but the increment in R^1 is smaller than that in X_1). These two assumptions can be rationalized as being a direct consequence of the second law of thermodynamics. Although the main theoretical results in this paper do not need any assumption on the derivative R_2, we stick to the most plausible case, which is $R_2 \leq 0$, and the most interesting results arise when $R_2 < 0$, meaning that the use of resource 2 makes waste classification more difficult, obstructs recycling and causes the recovered quantity R^1 to decrease. In the example of the boxes, we can never get more paper from recycling than the amount that was originally present as a component of the final product. The larger the amount of plastic that is present in the residuals, the more severe the mixing of waste and, as a consequence, the more difficult it becomes to separate and sort the paper, so recycling becomes less effective.

The natural stock of resource 1 evolves through time according to the state equation

$$\dot{S}_1(t) \equiv \frac{dS_1(t)}{dt} = g_1(S_1(t)) - X_1(t) + R(X_1(t), X_2(t)).$$

Resource 2 is not recyclable[2], so that $\dot{S}_2(t) = g_2(S_2(t)) - X_2(t)$. To simplify the notation, the time variable t is omitted when no ambiguity exists.

A social planner has the objective of maximizing the consumer's total discounted utility, so that he

[2]It is straighforward to include the possibility of recycling resource 2 by means of a function $R^2 = \Gamma(X_1, X_2)$. Nevertheless, this generalization adds little conceptual content and makes the mathematics much more tedious.

or she solves the problem

$$
\begin{aligned}
&\underset{\{X_1,X_2\}}{Max} \int_0^\infty U(Y)e^{-\delta t}dt \\
&s.t.: \\
&\quad Y = F\left(X_1, X_2\right), \\
&\quad \dot{S}_1 = g_1(S_1) - X_1 + R(X_1, X_2), \\
&\quad \dot{S}_2 = g_2(S_2) - X_2, \\
&\quad \left.\begin{aligned} &S_i(0) = S_i^0, \\ &S_i \geq 0, \end{aligned}\right\} \quad i = 1, 2,
\end{aligned} \tag{P}
$$

δ being the time discount rate and S_i^0 the initial stock of resource i, which is exogenously given. We will focus on interior solutions, that is, $X_i > 0$ holding throughout the solution.

(P) is an infinite horizon, continuous time, optimal control problem with two state variables and two control variables. Note that it resembles a neoclassical optimal economic growth model with two activity sectors, each one exploiting a different natural resource, where the stocks of both resources play the role of productive capital stocks, the natural growth functions g_i play the role of two sector production functions and the recycling technology plays the role of a negative externality between both sectors.

3 Model without recycling

André and Cerdá (2001a) analyze a particular case of model (P) without recycling ($R\left(X_1, X_2\right) = 0$). It is useful to gather the main results in such a case as a benchmark to study the effect of recycling on the solution. The ratio x evolves in agreement with the following differential equation[3]:

$$
\frac{\dot{x}}{x} = \sigma\left[g_1'(S_1) - g_2'(S_2)\right], \tag{1}
$$

$\sigma \equiv \frac{d\log(X_1/X_2)}{d\log(MRTS)}$ being the elasticity of substitution of F and $MRTS = \frac{F_2}{F_1}$ the Marginal Rate of Technical Substitution between both resources. The optimal output path satisfies the following differential equation[4]:

$$
\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)}\left[\xi_1 g_1'\left(S_1\right) + \xi_2 g_2'(S_2) - \delta\right], \tag{2}
$$

[3]André and Cerdá (2001a), Proposition 1.

[4]André and Cerdá (2001a), Proposition 2.

where $\eta(Y)$ is the intertemporal substitution elasticity of the utility function U, given by $\eta(Y) = \frac{-U''(Y)Y}{U'(Y)} \geq 0$, and ξ_1, ξ_2 are the returns to the i-th input, given by

$$\xi_i = \frac{X_i F_i}{F(X_1, X_2)} \geq 0, \qquad i = 1, 2. \tag{3}$$

Let us look at the economic interpretation of these results: according to (1), the evolution of x is determined by an environmental component -the difference between the marginal growth of both resources- and a technological component -the elasticity of substitution of the production function, in such a way that x increases (decreases), or equivalently, that X_1 (X_2) grows faster than X_2 (X_1), if the marginal growth of resource 1 is larger (smaller) than that of resource 2. In addition, the higher the elasticity of substitution, the faster the response of x to a difference between g_1' and g_2'. Condition (2) can be interpreted as a version of the Keynes-Ramsey rule of a standard neoclassical optimal economic growth model, where $\xi_1 S_1 + \xi_2 S_2$ is the *stock of natural* capital and $\xi_1 g_1'(S_1) + \xi_2 g_2'(S_2)$ plays the role of the *marginal productivity of natural capital.*[5]

When both resources are nonrenewable, equation (1) shows that $\dot{x} = 0$ and x remains constant throughout the solution and both resources are employed in a constant proportion or, equivalently, the use of both resources increases (or decreases) at the same rate. Specifically, x is given by[6]

$$x = \frac{\xi_1}{\xi_2} \Lambda \tag{4}$$

where $\Lambda = \frac{\lambda_2}{\lambda_1}$, λ_i being the costate variable associated with the stock of resource i. Note that $\dfrac{\xi_1}{\xi_2}$ measures the relative weight of both resources on production and Λ is a (relative) measure of the social valuation of both resources, that remains constant throughout the solution. Equation (2) states that $\frac{\dot{Y}}{Y} = \frac{-\delta}{\eta(Y)} < 0$ and the assumption of constant returns to scale implies $\frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2} = \frac{-\delta}{\eta(Y)}$.

If resource 2 is renewable and resource 1 is nonrenewable, then (1) becomes $\frac{\dot{x}}{x} = -\sigma g_2'(S_2)$ and, if $g_2'(S_2) > 0$, then $\dot{x} < 0$ and therefore the renewable resource tends to be more and more intensively used with respect to the nonrenewable resource. Equation (2) becomes $\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)} \left[\xi_2 g_2'(S_2) - \delta\right]$, according to which, output increases more (or decreases less) when the marginal growth of the renewable resource is larger and such a resource has a larger weight in the production technology, with respect to the discount rate.

[5] In equations (1) and (2), note that, in general, σ depends on x and $\eta(Y)$ depends on Y. Nevertheless, this is not the case for the production and utility functions which are traditionally used in economics. Typically F and U are such that both σ and η are constant.

[6] André and Cerdá (2001a), Proposition 4.

4 Solution of the Model with Recycling and Economic Implications

Substituting the production function in the objective functional of problem (P), the current value Hamiltonian is defined as

$$\mathcal{H}(S_1, S_2, X_1, X_2, \lambda_1, \lambda_2) = U[F(X_1, X_2)] + \lambda_1 [g_1(S_1) - X_1 + R(X_1, X_2)] + \lambda_2 [g_2(S_2) - X_2],$$

where λ_i is the costate variable associated with resource i, which can be interpreted as the social valuation of a further unit of the stock of resource i or, equivalently, the social cost of extracting one unit of such a resource. Finally, we can also interpret λ_i as a measure of scarcity of the resource i. Together with the state equations, the Maximum Principle necessary conditions for an interior solution are

$$U' F_1 = \lambda_1 (1 - R_1), \tag{5}$$

$$U' F_2 = \lambda_2 - \lambda_1 R_2, \tag{6}$$

and

$$\dot{\lambda}_i = \lambda_i (\delta - g_i'(S_i)), \tag{7}$$

$$\text{with} \quad \lim_{t\to\infty} e^{-\delta t} \lambda_i \geq 0, \quad \lim_{t\to\infty} e^{-\delta t} (\lambda_i S_i) = 0, \qquad i = 1, 2.$$

In equations (5) and (6) we can see the impact of recycling on the optimal solution. Both equations state the equality between the marginal utility and the marginal cost of using both natural resources. The marginal utility obtained by the consumer from extracting resource i (left hand side of (5) and (6)) is measured by the marginal utility of consumption times the marginal productivity of resource i. The marginal cost of using resource 1 (right hand side of (5)) equals the social valuation of maintaining such a resource for its future use (as measured by its shadow price λ_1) times $(1 - R_1)$, that represents the effective reduction of the stock. Recycling makes resource 1 effectively more abundant and hence its extraction becomes less expensive from the social point of view. In a hypothetical, extreme case with $R(X_1, X_2) = X_1$, in which recycling would allow the whole amount of resource 1 to be recovered, we have $R_1 = 1$ and the social cost of extracting resource 1 would equal zero, meaning that a 100% effective recycling technology is equivalent to an unbounded resource abundance. In the opposite case, when $R_1 = 0$, we obtain the same condition as in the non-recycling case, $U' F_1 = \lambda_1$.

The marginal cost of extracting resource 2 is measured as the shadow price of resource 2, λ_2, plus the negative effect on the recovery of resource 1 which, in turn, equals the shadow price of resource 1 (λ_1) times the marginal impact of resource 2 on the recyclability of resource 1 (R_2).

Note that, even if the beneficial environmental impacts of recycling are not explicitly modeled, the simple fact that a recycling technology is available is likely to cause some market failures. Typically, recycling markets are not perfect. Assume, in an extreme case, that they are nonexistent or, equivalently, that there is no relationship between the virgin inputs market and the recycling products market, in such a way that the resource owners and the conventional producers do not take into account that, after consumption, the waste stream can be recycled, so their production decisions are made as if recycling did not exist. In that case, the private optimality conditions are given by

$$U^{'}F_i = \lambda_i\,, \qquad i = 1, 2 \tag{8}$$

where, in a competitive equilibrium, the private price for resource i would equal the shadow price λ_i. A social planner would be aware that the effective value of the resources is different from the market price, and so some policy instrument should be implemented to correct the market prices Note that the efficient solution can not be obtained, in general, with a single policy instrument and we need at least two. Specifically, the price of resource 1 should be corrected downwards by the amount $\lambda_1 R_1$. This can be done by a per-unit subsidy equal to $\lambda_1 R_1$ or, equivalently, an ad-valorem subsidy equal to R_1. The price of resource 2 should be increased by the amount $-\lambda_1 R_2$ (recall that $R_2 \leq 0$), which can be performed, for example, with a per-unit tax equal to $-\lambda_1 R_2$ or, equivalently, with an ad-valorem tax equal to $\frac{-\lambda_1 R_2}{\lambda_2}$.

Assume that the owners of resource 1 become aware of the possibility of recycling their own resource and they implement some efficient waste collection mechanism to take advantage of this possibility or, equivalently, some efficient secondary material market arises, so that the recyclability of resource 1 is correctly internalized. In this case, the market outcome will guarantee the fulfillment of condition (5), but the externality caused by the use of resource 2 on the recyclability of resource 1 still remains, so that condition (6) is not likely to hold. Obviously, if the amount of resource 2 does not affect the recyclability of resource 1, the latter effect does not show up, so a perfect market for the recycled material 1 is enough to ensure overall efficiency.

In the most general case, in this setting there are at least two sources of market failure that require two policy instruments. Note also that, given that R_1 and R_2 are functions of X_1 and X_2 or, more precisely, of x (given the homogeneity assumption), so is the size of the externalities and, as far as x is time-varying, the value of the required policy instruments should be also time varying. This observation is in the same spirit as Sinclair (1992) and other authors, noting that a constant tax is not generally effective when dealing with exhaustible resources.

Nonetheless, the main focus of the present paper is not on the specific instruments that should be implemented to correct these externalities, but on the nature of the externalities themselves and their economic dynamic implications. According to this aim, the rest of the paper is mainly devoted to showing the different dynamic nature of the solution for the recycling model as compared with the situation without recycling.

Sufficient optimality conditions for problem (P) are given in proposition 1.

Proposition 1 *In an interior solution for problem (P), the Arrow sufficient conditions for global maximum hold if and only if, throughout the solution,* $\lambda_1, \lambda_2 \geq 0$ *holds.*

Proof: See section 8.1

In agreement with proposition 1, a solution satisfying the Maximum Principle conditions is a global maximum for problem (P), iff $\lambda_1, \lambda_2 \geq 0$. Given the assumptions on U, F and R, we obtain from (5) and (6) that $\lambda_1 \geq 0$ trivially holds, but $\lambda_2 \geq 0$ requires the additional technical assumption

$$F_2 + R_2 \frac{F_1}{1 - R_1} \geq 0, \tag{9}$$

meaning that the marginal productivity of resource 2 in production is larger than its marginal negative effect on recycling.

For being F and R homogeneous of degree 1, we are allowed to define the following functions depending just on x:

$$\frac{F(X_1, X_2)}{X_2} = F\left(\frac{X_1}{X_2}, 1\right) \equiv f(x), \qquad \frac{R(X_1, X_2)}{X_2} \equiv r(x). \tag{10}$$

Propositions 2 and 3 are the main results for problem (P) and state the optimal temporal evolution of the relative use of resources and the output path.

Proposition 2 *In an interior solution for problem (P), the temporal evolution of* x *is given by the following differential equation:*

$$\frac{\dot{x}}{x} = \hat{\sigma}\left[g_1'(S_1) - g_2'(S_2)\right], \tag{11}$$

where

$$\hat{\sigma} \equiv \frac{-f' \left[f \left(1 - r' \right) + f' \left(r - x \right) \right]}{x \left[f f'' \left(1 - r' \right) + f f' r'' \right]}, \tag{12}$$

$f \equiv f(x)$ *etc.*

Proof: See section 8.2

Proposition 3 *In an interior solution for problem (P), the temporal evolution of output is ruled by the following differential equation:*

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)} \left[\hat{\xi}_1 g_1' (S_1) + \hat{\xi}_2 g_2' (S_2) - \delta \right], \tag{13}$$

where

$$\eta(Y) \equiv \frac{-U''(Y) Y}{U'(Y)} \qquad \hat{\xi}_1 \equiv \frac{f'(x-r)}{f(1-r')} \qquad \hat{\xi}_2 \equiv \frac{f(1-r') - f'(x-r)}{f(1-r')} \tag{14}$$

and $\hat{\xi}_1 + \hat{\xi}_2 = 1$ *holds.*

Proof: See section 8.3

Equation (11) generalizes equation (1) and has a similar interpretation: throughout the solution, the evolution of x depends on two factors: the difference between the marginal growth of both resources and the technological coefficient $\hat{\sigma}$. Such a coefficient plays a similar role to that of the elasticity of substitution of F in the model without recycling, i.e. a measure of technological flexibility, except for the fact that the technology has two components in model (P): production and recycling. The complex analytical expression of $\hat{\sigma}$ prevents us from clearly distinguishing the role of both components. Section 6 provides a way to enlighten this interpretation.

Equation (13) is a version of the Keynes-Ramsey rule where $\hat{\xi}_1 S_1 + \hat{\xi}_2 S_2$ plays the role of the *stock of natural capital* and $\hat{\xi}_1 g_1' + \hat{\xi}_2 g_2'$ measures the marginal productivity of natural capital, in such a way that output increases or decreases depending on the difference between the *marginal productivity of natural capital* and the discount rate. The difference between (13) and (2) lies in the weight of resources on *natural capital*. In (2) ξ_i measures the participation of resource i in production. In (13) $\hat{\xi}_i$ measures the participation of resource i in the aggregate technology, now including production and recycling. Section 6 provides an alternative way to interpret the coefficients in (14).

Note that recycling may produce two different effects on the solution: First, as shown in Weinstein and Zeckhauser (1974), the recyclable resource becomes less scarce, in such a way that recycling is similar to an exogenous increment of the resource stock. We label this impact as *scale effect.* Furthermore, recycling changes the technological framework of the economy by altering the value of $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{\sigma}$, and hence the optimal temporal adjustment of output and resource use. We label this as *technological effect.* If we think of x and Y as functions of time, the first effect is basically a parallel shift of such functions, whereas the second effect changes their shape. If the recycling function has the linear form $R(X_1, X_2) = \beta X_1$, with $0 < \beta < 1$, then, the only technological interaction between both resources happens by means of the production function F, it turns out that $\hat{\sigma} = \sigma$, $\hat{\xi}_1 = \xi_1$, $\hat{\xi}_2 = \xi_2$ and only the *scale effect* appears.

Given that the technological effect of recycling shows up in the coefficients $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{\sigma}$, from equations (11) and (13) we can conclude that this effect becomes relevant when some of the resources is renewable. To get some further insight, let us compare now two possible situations: the first one with two nonrenewable resources, and the second one with a renewable and a nonrenewable resource.

When both resources are nonrenewable, in agreement with equation (11), throughout the solution of (P), $\dot{x} = 0$ holds and x remains constant. Its specific value is given in proposition 4.

Proposition 4 *In an interior solution for problem (P), when both resources are nonrenewable, the optimal value for the ratio x can be expressed as*

$$x = \psi \Lambda, \tag{15}$$

Λ and ψ being two coefficients that remain constant and are defined as

$$\Lambda \equiv \frac{\lambda_2}{\lambda_1}, \quad \psi \equiv \frac{X_1 F_1}{X_2 F_2 - R_1 F + F_1 R} \equiv \frac{x f'}{f(1 - r') + f'(r - x)}.$$

where $F_1 \equiv F_1(X_1, X_2)$ and so on.

Proof: See section 8.4 ¥

According to proposition 4, x comes from two factors: ψ, which is a measure of the relative technological weight of both resources, and Λ, which is a measure of social relative cost, or equivalently, of relative scarcity. If we compare this result with the equivalent for the nonrenewable case (4), we see that the ratio $\frac{\xi_1}{\xi_2}$, measuring the relative share of both resources in production, is replaced by the new ratio ψ, measuring the relative share of both resources in the new aggregate technology (encompassing conventional production and recycling).

From (13) we know that (as in the non-recycling case) $\frac{\dot{Y}}{Y} = \frac{-\delta}{\eta(Y)}$. The homogeneity assumption on F

implies $\frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2} = \frac{-\delta}{\eta}$ as well[7], in such a way that the output and the instantaneous extraction of both resources continuously decrease through time at a rate equal to $\frac{-\delta}{\eta(Y)}$.

Observe that, in the case with two non-renewable resources, we get the same qualitative dynamic properties of the solution as in the no-recycling case: x remains constant for ever and Y steadily decreases at a constant rate equal to $\frac{-\delta}{\eta(Y)}$. As a consequence, only the scale effect shows up. If a recycling technology suddenly becomes available, then the value of the ratio x should be adjusted once and for all by some parallel shift (presumably upwards). The larger availability of resource X_1 will also imply an upwards shift of the output path, but the shape of the relevant variables is not affected.

Assume that **resource 1 is recyclable and nonrenewable whereas resource 2 is renewable and non-recyclable**. Then equation (13) becomes

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta\left(Y\right)}\left[\hat{\xi}_2 g_2'\left(S_2\right) - \delta\right], \tag{16}$$

so that production may increase or decrease through time depending on the marginal growth of resource 2, the weight of such resource on the technology, as measured by $\hat{\xi}_2$, and the discount rate. Provided that $\eta\left(Y\right) > 0$, Y turns out to be the more increasing (or the less decreasing), the larger the marginal growth of the renewable resource and its weight on technology. Note that the long term evolution of output is determined by the properties of the renewable resource. For smaller values of the marginal growth and the parameter $\hat{\xi}_2$, the case with one renewable resource is *more similar* to that with two nonrenewable resources and output decreases faster (or increases slower). Equation (11) takes the form

$$\frac{\dot{x}}{x} = -\hat{\sigma} g_2'\left(S_2\right), \tag{17}$$

and the evolution of x depends just on the marginal natural growth of resource 2 and the technological coefficient $\hat{\sigma}$. If such a coefficient is positive and throughout the solution $g_2'\left(S_2\right) > 0$ holds, then, x continuously decreases with time and output depends more and more heavily on the renewable resource, in the same fashion as in the no-recycling case.

The use of renewable resources and recyclable resources are two strategies claimed to be relevant for economic sustainability. Insofar as we do not explicitly model some key elements such as physical capital accumulation and technological change, we can not draw very general results concerning sustainability

[7] Deriving the equation $Y = F\left(X_1, X_2\right)$ with respect to time, we obtain $\dot{Y} = F_1\dot{X}_1 + F_2\dot{X}_2$. Dividing both sides by Y and $F\left(X_1, X_2\right)$ respectively and using (3), we know that $\frac{\dot{Y}}{Y} = \xi_1\frac{\dot{X}_1}{X_1} + \xi_2\frac{\dot{X}_2}{X_2}$; $\frac{\dot{x}}{x} = 0$ (see section 4) implies $\frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2}$, and $\xi_1 + \xi_2 = 1$ (since F is homogeneous of degree 1) implies $\frac{\dot{Y}}{Y} = \frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2}$.

from our model. Nevertheless, we can obtain some insight about the contribution of recyclable and renewable resources to the achievement of sustainable solutions.

The idea of sustainability in economics is linked to the possibility of obtaining a non-decreasing time output path. Following this idea we can say, stretching the meaning of the terms somewhat, that, comparing two decreasing paths, if one of them is more sharply decreasing than the other, then, the latter belongs to a more sustainable economy than the former. From equations (16) and (17) we can conclude that recycling increases the availability of resources and affects the temporal evolution pattern of output and resource use. Nevertheless, the sign of the output increase or decrease mainly depends on the natural growth ability of resources. Henceforth, as regards sustainability, recycling is a useful and relevant short term strategy but, in the long run, the possibility of obtaining sustainable paths crucially depends on the extent to which production rests more and more intensively on renewable resources. This conclusion can be seen as a consequence of the second law of thermodynamics. As far as recycling is not "perfect" (in the sense that some material and energy is always lost in the process), it is not capable, by itself, offsetting the depletion of exhaustible resources.

5 Example

In the previous section we have shown that the presence of a recycling technology may crucially affect the shape of the production set of the economy and so, apart from increasing the effective availability of the recyclable resources, it is likely to have an important impact on the shape of the dynamic paths for output and natural resource use. Nevertheless, with this level of generality it is difficult to get some clear-cut results about the specific dynamic impact of recycling. In this section we try to get some further insight by discussing the solution with specific production and recycling functions, so that we can illustrate the relevance of recycling for the dynamic behavior of the variables of interest.

Assume that the production function is of the Cobb-Douglas type $F(X_1, X_2) = X_1^{\alpha_1} X_2^{\alpha_2}$, with $\alpha_1 + \alpha_2 = 1$, and the utility function is $U(Y) = \frac{Y^{1-\eta}}{1-\eta}$ with constant elasticity of temporal substitution equal to $\eta \in (0, 1)$.

We compare the no-recycling case[8] with a case where the recycling technology is given by the following

[8] All the results for the no-recycling case shown in this section are proved in André and Cerdá (2001a).

function:

$$R^1 = R(X_1, X_2) = \frac{X_1^2}{X_1 + X_2} = X_1 \cdot h(x), \tag{18}$$

where $h(x) \equiv \frac{x}{1+x}$, satisfying $h(0) = 0$, $\lim_{x\to\infty} h(x) = 1$, $0 < h' = \frac{1}{(1+x)^2} < 1$, $h'' = \frac{-2}{(1+x)^3} < 0$, $\forall \quad x > 0$. According to (18), for given values of X_1 and X_2, recycling allows a proportion h of resource 1 to be recovered, h positively depending on the value of the ratio x. Note that $X_2 = 0$ implies $R^1 = X_1$, meaning that the whole amount of resource 1 can be recovered through recycling. This situation is equivalent to an unlimited abundance of resource 1[9]. Nevertheless, given that both resources are essential for production, the situation $X_1 > 0$, $X_2 = 0$ never happens in the optimal solution.

Two nonrenewable resources

Assume, first, that both resources are non-renewable. Also make the technical assumption $S_2^0 > S_1^0$. In the no-recycling case, the optimal extraction rate for each resource is given by

$$X_i = A_i e^{\frac{-\delta t}{\eta}} \qquad i = 1, 2, \qquad \text{where} \quad A_i = \frac{\delta S_i^0}{\eta} > 0 \quad i = 1, 2. \tag{19}$$

Substituting the expressions for X_1 and X_2 in the production function, we obtain the output path

$$Y = A_1^{\alpha_1} A_2^{1-\alpha_1} e^{\frac{-\delta t}{\eta}} = \frac{\delta S_2^0}{\eta} \left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1} e^{\frac{-\delta t}{\eta}}, \tag{20}$$

and dividing X_1 by X_2,

$$x \equiv \frac{X_1}{X_2} = \frac{S_1^0}{S_2^0}. \tag{21}$$

Note that X_1, X_2 and Y decrease over time at a constant rate $\frac{\delta}{\eta}$. As shown in the theoretical results, the relative input intensity x remains constant and (21) shows that its value is given by the initial stock of both resources. Note that it does not depend on the value of α_1.

Assume now that resource 1 can be recycling according to the technology described in (18). As shown in section 8.5, the solution for X_1 and X_2 is given by $X_1 = \frac{\delta S_1^0 S_2^0}{\eta(S_2^0 - S_1^0)} e^{-\frac{\delta}{\eta}t}$, $X_2 = \frac{\delta S_2^0}{\eta} e^{-\frac{\delta}{\eta}t}$. Therefore, the relative use of resources takes the form $x = \frac{S_1^0}{S_2^0 - S_1^0}$ and output becomes $Y = \frac{\delta}{\eta} S_2^0 \left(\frac{S_1^0}{S_2^0 - S_1^0}\right)^{\alpha_1} e^{-\frac{\delta}{\eta}t}$.

In the absence of renewable resources, even if the possibility of recycling exists, the use of both resources and output decreases at the rate $\frac{\delta}{\eta}$, the same as in the no-recycling case. Note, nevertheless, the different impact of the initial resource stocks on the solution. For the non-recycling case, the following

[9] A slightly more general specification not involving this extreme possibility is $R^1 = \frac{bX_1^2}{X_1 + X_2}$ with $b \in (0, 1)$.

sensitivity analysis results are easy to show:

$$\frac{\partial X_i}{\partial S_j^0}\begin{cases} >0 & if \quad i=j \\ \\ =0 & if \quad i\neq j \end{cases} \qquad\qquad \frac{\partial Y}{\partial S_i^0}>0, \qquad i=1,2.$$

The whole optimal path of the extraction rate of a resource depends positively on the initial stock of the same resource and does not depend at all on the initial stock of the other resource. Output positively depends on the stock of both resources. The following table shows the parallel signs for the case with recycling[10]:

	X_1	X_2	Y
S_1^0	$+$	0	$+$
S_2^0	$-$	$+$	$?$

$$\frac{\partial Y}{\partial S_2^0}\geq 0 \Leftrightarrow \frac{S_2^0}{S_1^0}\geq \frac{1}{(1-\alpha_1)}.$$

As in the non-recycling case, if S_1^0 increases, the whole path of X_1 shifts upwards and that of X_2 remains unchanged, so that the path of Y unambiguously shifts upwards. Nevertheless, unlike the non-recycling case, the instantaneous use of resource 1 diminishes with S_2^0 because of the negative technological interaction between both resources by means of recycling. If resource 2 becomes more abundant, it is more intensively used, negatively affecting the recycling recovery of resource 1, causing an effective shortening of the availability of resource 1. The resulting effect on Y is ambiguous and depends on the initial relative availability of both resources.

One renewable and one non-renewable resource

The most interesting case arises when there is one renewable resource. We will show that, in this case, the dynamics of the solution is much more complex when a recycling technology exists. To keep the discussion as simple as possible, assume that resource 1 is nonrenewable ($g_1(S_1)=0$) and resource 2 is renewable with a constant growth rate, so that $g_2(S_2)=\gamma_2 S_2$. Make the technical assumptions also $\delta>\alpha_2\gamma_2(1-\eta)$ to ensure the existence of solution.

In the no-recycling case, the solution for X_1 and X_2 is given by

$$X_i = K_1 S_i^0 e^{-K_i t}>0, \tag{22}$$

[10]Note that these results apply for "small" changes in S_1^0 and S_2^0, in such a way that the technical assumption $S_1^0<S_2^0$ still holds.

where $K_1 = \frac{\delta - \alpha_2\gamma_2(1-\eta)}{\eta} > 0$, $K_2 = \frac{\delta - \gamma_2[1-\alpha_1(1-\eta)]}{\eta} \lessgtr 0$, so that X_1 asymptotically decreases to zero, whereas X_2 may increase or decrease depending on the sign of K_2. Dividing the X_1 equation by the X_2 equation, we obtain $x = \frac{X_1}{X_2} = \frac{S_1^0}{S_2^0} e^{-\gamma_2 t}$ and substituting (22) in the production function, we obtain the output path

$$Y = \left(S_1^0\right)^{\alpha_1} \left(S_2^0\right)^{1-\alpha_1} \frac{\left[\delta - (1-\alpha_1)\gamma_2(1-\eta)\right]}{\eta} e^{\frac{(1-\alpha_1)\gamma_2 - \delta}{\eta} t} > 0. \tag{23}$$

So, in this case the renewable resource replaces the nonrenewable one at a constant rate γ_2. Output increases or decreases (depending on the sign of $(1-\alpha_1)\gamma_2 - \delta$) at a constant rate $\frac{(1-\alpha_1)\gamma_2 - \delta}{\eta}$. If the share of the renewable resource (α_2) is large enough, or its growth ability (as measured by γ_2) is large enough, as compared with the discount rate δ, then it would be possible to have a balanced growth path with a positive constant growth rate of output. Otherwise, output monotonically decreases and goes asymptotically to zero.

Assume now that resource 1 can be recycled according to (18). Equations (16) and (17) become

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta}\left[\hat{\xi}_2\gamma_2 - \delta\right] \qquad \text{and} \tag{24}$$

$$\frac{\dot{x}}{x} = -\hat{\sigma}\gamma_2. \tag{25}$$

In this case, it is not possible to find an analytical solution and we present an analysis of the qualitative behavior of the solution. Since both resources are essential for production[11], and $U' > 0$ we can discard corner solutions with $X_1 = 0$ or $X_2 = 0$, and concentrate on interior solutions. Substituting the expressions for f and r in (12), we obtain the following expression for coefficient $\hat{\sigma}$:

$$\hat{\sigma} = \frac{(1+x)^2\alpha_1 - (1+x)}{(\alpha_1 - 1) + (\alpha_1 + 1)x}. \tag{26}$$

Observe that $\lim_{x\to 0}\hat{\sigma} = 1$, $\lim_{x\to\infty}\hat{\sigma} = \infty$ and $\hat{\sigma}$ has a vertical asymptote at the point $x = \frac{1-\alpha_1}{1+\alpha_1}$, as illustrated in Figure 1.

INSERT FIGURE 1

Given (25) and the shape of $\hat{\sigma}$, we can study the optimal substitution between both resources depending on the value of x: if $x \in \left[0, \frac{1-\alpha_1}{1+\alpha_1}\right)$, then $\hat{\sigma} > 0$ and x decreases to zero. If $x \in \left(\frac{1-\alpha_1}{1+\alpha_1}, \frac{1-\alpha_1}{\alpha_1}\right)$, then $\hat{\sigma} < 0$ and, according to (25) x tends to increase until $x = \frac{1-\alpha_1}{\alpha_1}$, a point in which $\dot{x} = \hat{\sigma} = 0$. If $x > \frac{1-\alpha_1}{\alpha_1}$,

[11] i.e. $F(X_1, 0) = F(0, X_2) = 0$. See, for example, Hartwick (1978a).

then $\hat{\sigma} > 0$ and x diminishes towards $x = \frac{1-\alpha_1}{\alpha_1}$. If the solution path reaches the value $x = \frac{1-\alpha_1}{\alpha_1}$, then $\hat{\sigma} = 0$, and (25) implies $\dot{x} = 0$; furthermore $\hat{\xi}_1 = 1$, $\hat{\xi}_2 = 0$, (16) becomes $\frac{\dot{Y}}{Y} = \frac{-\delta}{\eta}$, and the evolution of x and Y is identical to that of a model with two nonrenewable resources.

Substituting the expressions for f and r in (14) and rearranging we conclude that $\hat{\xi}_1$ and $\hat{\xi}_2$ are, in this case, linear functions of x: $\hat{\xi}_1 = \alpha_1 (1 + x)$, $\hat{\xi}_2 = 1 - \alpha_1 (1 + x)$. Note that, at the point $x = 0$, we have $\hat{\xi}_1 = \xi_1 = \alpha_1$ and $\hat{\xi}_2 = \xi_2 = 1 - \alpha_1$, so that the weight of both resources in the aggregate technology is the same as in the conventional technology without recycling. The weight of resource 1 (2) increases (decreases) with x. From (24), we can infer the evolution of output throughout the solution. Assume first $\delta > (1 - \alpha_1)\gamma_2$. In this case, Y decreases for every value of x (as in the no-recycling case). Furthermore, (24) implies that, the larger the value of x (implying that production rests more heavily on the nonrenewable resource), the faster output decreases.

Figure 2 compares the phase diagram for the solutions with and without recycling.

INSERT FIGURE 2

Recall that, in the no-recycling case, the nonrenewable resource is substituted by the renewable one at a constant rhythm, in such a way that x decreases at a rate equal to γ_2, and output decreases at a constant rate $\frac{\delta - \alpha_2 \gamma_2}{\eta}$. In the recycling case, output still decreases but the rate of decay depends on the value of x and so, it is changing throughout the solution. Concerning the substitution of natural resources, there are two cases depending on the initial conditions for S_1 and S_2. There is one case, similar to the no-recycling one in which the renewable resource still replaces the nonrenewable one over time so that x tends to zero. Nevertheless, there is also a rest point at $\bar{x} \equiv \frac{1-\alpha_1}{\alpha_1}$ in such a way that x can increase or decrease as it converges to $\bar{x}$. Intuitively, when changing the value of x, there is a trade-off between the growing ability of the renewable resource, which contributes to x being decreasing, and the technological resource interaction through recycling, which prevents x from decreasing too much (remember that increasing X_2 decreases the recycling ability of resource 1). From Figure 2, we can conclude that, in the latter case, the technological interaction effect is important enough to make it optimal to keep a positive ratio of both resources in the long run, despite the renewable ability of the second resource.

Consider now the situation $\delta < (1 - \alpha_1)\gamma_2$. If there is no recycling, according to (23), output grows over time at a constant rate equal to $\frac{(1-\alpha_1)\gamma_2 - \delta}{\eta}$. In the recycling case, according to (24), Y increases (decreases) if x is smaller than (larger than) the threshold value $\frac{(1-\alpha_1)\gamma_2-\delta}{\alpha_1\gamma_2}$. Figure 3 compares the dynamic evolution of the solution in both cases[12].

INSERT FIGURE 3

Once again, note that, in the recycling case there is a range for the initial conditions such that the behavior of the solution is analogous to that of the no-recycling case: the ratio x tends to zero and output growths unlimitedly. In this case, the growth rate of Y increases as x decreases and it tends asymptotically to $\frac{(1-\alpha_1)\gamma_2-\delta}{\eta}$. Nevertheless, there is still a rest point at $\bar{x} = \frac{1-\alpha_1}{\alpha_1}$, so that x increases or decreases as it converges to $\bar{x}$. Interestingly, Y may display an inverted U shape for some initial conditions. If x is 'low' (but not 'very low', so that we are not in the first region), the growth rate of Y is positive, according to (24). As x increases, the share of the renewable resource decreases, which affects negatively the growth rate of Y, up to a point where $\frac{\dot{Y}}{Y}$ becomes zero and then negative, as x keeps on increasing.

6 An integrated view of production and recycling: the Production and Recycling Function (PRF)

The production side of the economy is shaped by numerous forces including the technological structure and the decisions of all the active firms as well as all the interactions among them. Nevertheless, what matters to assess the consumption and production decisions, from a macroeconomic point of view, is the aggregated production set of the economy, as represented by the production function of the economy F, that can be seen as a reduced form of all these individual forces. Along this article, we have shown the technological relevance of recycling, which alters the technological framework and the production set of the economy. As a consequence, recycling turns out to be one of the elements that should be taken into account as an important component of the overall technological structure of the economy.

In this section, we look for an instrument, equivalent to the production function, that could represent the new production set once recycling has been incorporated. For that purpose, we introduce the new

[12]The diagram for the recycling case is drawn under the combination

$$\frac{\gamma_2(1-\alpha_1)}{1+\alpha_1} > \delta > \gamma_2(1-\alpha_1)(1-\eta)$$

which is feasible if η is large enough.

concept of Production and Recycling Function (PRF), which is a generalization of the production function, aimed at representing the new production set of the economy when recycling is taken into account. This concept provides an alternative way to formulate problem (P) and interpret its solution. The main idea is to take full advantage of the economic interpretation given in section 3 for the problem without recycling.

Denoted by Z_1 and Z_2, let the *net extraction rate* of resources 1 and 2 be defined as

$$Z_1 = X_1 - R\left(X_1, X_2\right), \tag{27}$$

$$Z_2 = X_2, \tag{28}$$

measuring the instantaneous effective reduction of the stock of each resource taking into account the quantity extracted for production and the quantity recovered from recycling[13]

The Implicit Function Theorem guarantees that equation (27) can be locally solved for X_1 by a implicitly defined, $C^{(2)}$ function ϕ, in such a way that

$$X_1 = \phi\left(Z_1, Z_2\right). \tag{29}$$

Substituting (27), (28) and (29) in the production function we obtain an expression for Y as a function of Z_1 and Z_2: $Y = F\left(X_1, X_2\right) = F\left(\phi\left(Z_1, Z_2\right), Z_2\right) = \tilde{F}\left(Z_1, Z_2\right)$, so that the original problem can be reformulated as

$$\begin{aligned}
&\underset{\{Z_1, Z_2\}}{Max} \int_0^\infty U(Y)e^{-\delta t}dt \\
&s.t.: \\
&\quad Y = \tilde{F}(Z_1, Z_2), \\
&\quad \dot{S}_i = g_i(S_i) - Z_i, \quad i = 1, 2 \\
&\quad S_i(0) = S_i^0, \quad i = 1, 2, \\
&\quad S_i, Z_i \geq 0, \quad i = 1, 2.
\end{aligned} \tag{P'}$$

We can rationalize problem (P') in the following way: assume that, in order to produce Y, apart from the amount $X_2 \equiv Z_2$ of resource 2, we use the amount X_1 of resource 1, coming from two sources: recycled material R^1, and virgin material Z_1. Furthermore, assume that the recycled material R^1 is always immediately reincorporated into the production process, so that, using (27), we can obtain the instantaneous amount of required virgin material Z_1, as the difference between total requiered input and the available recycled material. The assumptions on R guarantee that $Z > 0$.

[13] If resource 2 were assumed to be recyclable, then Z_2 should be defined in the same way as Z_1.

Note that (P) and (P') contain the same elements, therefore they are fully equivalent. From the solution to (P'), that of (P) can be obtained by undoing the variable change given in (27) and (28) and vice versa. The auxiliary variables Z_i allow us to include the recycling technology, not as a part of the state equations, but as a part of the technology, represented by $\tilde{F}$.

Figure 4 compares graphically both alternative formulations: In (P), both types of technology are represented separately by means of the functions F and R. Recycling alters the production set but, taking the formulation (P) into account, it is not possible to express such an effect in a compact way. Formulation (P') defines the new production set by a single function $\tilde{F}$, jointly considering both technologies.

INSERT FIGURE 4

Both resources 1 and 2 have a two-fold effect on output: a direct effect, through conventional production, and an indirect effect, by means of resource recovery through recycling. The recovered resource can be reused for production to obtain some additional output. For resource 1, both effects are positive, whereas for resource 2, the direct effect is positive and the indirect one is negative. The representation (P') could be directly applied to a problem without recycling, which is a particular case with $Z_1 = X_1$, $Z_2 = X_2$ and $\tilde{F} = F$. The PRF $\tilde{F}$ includes the aggregate effect determined by the direct one and the indirect one. The main mathematical properties of $\tilde{F}$ are the following:

i) $\tilde{F}$ is of class $C^{(2)}$ since it is a composition of $C^{(2)}$ functions.

ii) $\tilde{F}$ is homogeneous of degree 1, so that it presents constant returns to scale. Multiplying both sides of (27) by $\alpha > 0$, using (28) and the homogeneity assumption on R, we have $\alpha Z_1 = \alpha X_1 - \alpha \left[R\left(X_1, X_2\right)\right] = \alpha X_1 - \left[R\left(\alpha X_1, \alpha Z_2\right)\right]$. From the Implicit Function Theorem we know that $\alpha X_1 = \phi\left(\alpha Z_1, \alpha Z_2\right)$ and, since F is homogeneous of degree 1, $\tilde{F}\left(\alpha Z_1, \alpha Z_2\right) = F\left(\phi\left(\alpha Z_1, \alpha Z_2\right), \alpha Z_2\right) = F\left(\alpha X_1, \alpha X_2\right) = \alpha F\left(X_1, X_2\right) = \alpha \tilde{F}\left(Z_1, Z_2\right)$. The homogeneity of degree 1 allows us to define the function $\tilde{f}(z)$ given by $\frac{\tilde{F}(Z_1, Z_2)}{Z_2} = \tilde{F}\left(\frac{Z_1}{Z_2}, 1\right) = \tilde{F}(z, 1) = \tilde{f}(z)$, only depending on $z = \frac{Z_1}{Z_2}$, and we know that $\tilde{F}_1 = \tilde{f}'(z)$, $\tilde{F}_2 = \tilde{f}(z) - z\tilde{f}'(z)$.

iii) As for the first derivatives of $\tilde{F}$, substitute (28) and (29) in (27) to obtain

$$Z_1 = \phi\left(Z_1, Z_2\right) - R\left(\phi\left(Z_1, Z_2\right), Z_2\right). \tag{30}$$

Deriving (30) with respect to Z_1 and Z_2 by the chain rule and rearranging,

$$\phi_1 \equiv \frac{\partial \phi}{\partial Z_1} = \frac{1}{1-R_1} > 0, \qquad \phi_2 \equiv \frac{\partial \phi}{\partial Z_2} = \frac{R_2}{1-R_1} \leq 0. \tag{31}$$

Deriving $\tilde{F}$ with respect to Z_1 and Z_2, and using (31),

$$\tilde{F}_1 \equiv \frac{\partial \tilde{F}}{\partial Z_1} = \frac{\partial F}{\partial X_1}\frac{\partial \phi}{\partial Z_1} \equiv F_1\phi_1 = \frac{F_1}{1-R_1} \geq F_1 > 0, \tag{32}$$

$$\tilde{F}_2 \equiv \frac{\partial \tilde{F}}{\partial Z_2} = \frac{\partial F}{\partial X_2} + \frac{\partial F}{\partial X_1}\frac{\partial \phi}{\partial Z_2} \equiv F_2 + F_1\phi_2 = F_2 + F_1\frac{R_2}{1-R_1} \leq F_2. \tag{33}$$

The marginal productivity of resource 1 is positive and, if $R_1 > 0$, then $\tilde{F}_1 > F_1$, in such a way that the productivity of resource 1 in the aggregate technology is larger than that in the conventional production, because both effects are the same sign. For resource 2 the two effects have the opposite sign, so $\tilde{F}_2 \leq F_2$. The sign of $\tilde{F}_2$ could be ambiguous, in principle, but the technical assumption (9) implies $\tilde{F}_2 > 0$, so that the direct positive effect overcomes the negative indirect effect of resource 2 on output. If $R_2 < 0$, then $\tilde{F}_2 < F_2$. The Marginal Rate of Technical Substitution (MRTS) of $\tilde{F}$ turns out to be

$$\left|\frac{dZ_1}{dZ_2}\right|_{\bar{Y}} = \frac{\tilde{F}_2}{\tilde{F}_1} = \frac{F_1R_2 + F_2(1-R_1)}{F_1} = \frac{F_2}{F_1}(1-R_1) + R_2 < \frac{F_2}{F_1},$$

$\left|\frac{dZ_1}{dZ_2}\right|_{\bar{Y}}$ representing the slope, in absolute value, of an isoquant of $\tilde{F}$, which is smaller than that of an isoquant of F.

iv) $\tilde{F}$ includes F as a particular case when $R(X_1, X_2) = 0$, $Z_1 = X_1$, $Z_2 = X_2$, $\tilde{F}(Z_1, Z_2) = \tilde{F}(X_1, X_2) = F(X_1, X_2)$.

Define the relative effective extraction ratio as $z = \frac{Z_1}{Z_2}$. Provided that the mathematical formulation of (P') applies to the non-recycling case discussed in section 3, replacing $\tilde{F}$ by F and X_i by Z_i, both main equations for the problem without recycling, (1) and (2), are directly applicable to (P'). Specifically z evolves according to

$$\frac{\dot{z}}{z} = \tilde{\sigma}\left[g_1'(S_1) - g_2'(S_2)\right], \tag{34}$$

$\tilde{\sigma}$ denoting the elasticity of substitution of $\tilde{F}$ and having the traditional economic interpretation: the technological flexibility to substitute the use of resources while keeping output constant. In agreement with (34), if $\tilde{\sigma} > 0$, then provided that $g_1'(S_1) > g_2'(S_2)$, $\frac{\dot{z}}{z} > 0$ holds, then the net extraction rate of resource 1 increases faster than that of resource 2. Furthermore, the larger the elasticity of substitution of the PRF (i.e., the more flexible the aggregate technology) the faster the temporal adjustment of z in response to a difference between g_1' and g_2'. Equation (2) takes the following expression:

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)} \left[\tilde{\xi}_1 g_1' (S_1) + \tilde{\xi}_2 g_2' (S_2) - \delta \right], \tag{35}$$

$\tilde{\xi}_i = \frac{Z_i \tilde{F}_i}{\tilde{F}(Z_1,Z_2)}$ representing the returns of resource i on the PRF. If the sufficient conditions for global optimality hold, then we know that $\tilde{F}_1,\ \tilde{F}_2 > 0$, since $\tilde{F}$ is homogeneous of degree 1, $\tilde{\xi}_1 + \tilde{\xi}_2 = 1$ and $\tilde{\xi}_1 g_1' (S_1) + \tilde{\xi}_2 g_2' (S_2)$ turns out to be a linear convex combination of the marginal growth of both resources, that measures the *marginal productivity of natural capital* in problem (P'), where $\tilde{\xi}_1 S_1 + \tilde{\xi}_2 S_2$ measures the stock of natural capital. Condition (35) is a version of the Keynes-Ramsey rule stating that instantaneous output grows or diminishes depending on the difference between the natural capital productivity and the discount rate.

We can now draw the relationship between the solutions to (P') and (P). If (35) and (13) are to hold simultaneously, then $\hat{\xi}_i = \tilde{\xi}_i$ must hold at every instant. From the formulation (P') we know that $\tilde{\xi}_i$ (and hence $\hat{\xi}_i$) represents the weight of resource i on the PRF, providing an interpretation for the coefficients $\hat{\xi}_i$ that was not available in section 4. In order to find the relation between equations (34) and (11), note that, using the definition of z and x jointly with (27) and (28), we know that $z = x - r(x)$, and taking derivatives with respect to time we obtain

$$\dot{z} = \dot{x} - r'(x)\,\dot{x} = \dot{x}\,(1 - r'(x)), \tag{36}$$

from which, given that $r'(x) = R_1 \in [0,1)$, $\dot{z}$ and $\dot{x}$ have the same sign. Dividing both sides of (36) by z and $x - r(x)$ respectively, and using (34) to substitute $\dfrac{\dot{z}}{z}$, we have

$$\frac{\dot{x}}{x} = \frac{x - r(x)}{x\,(1 - r')} \tilde{\sigma} \left[g_1'(S_1) - g_2'(S_2) \right]. \tag{37}$$

Comparing (37) and (11), $\hat{\sigma}$ turns out to be a simple, sign-preserving transformation of the elasticity of substitution of the PRF: $\hat{\sigma} = \frac{x - r(x)}{x(1-r')} \tilde{\sigma} = \frac{X_1 - R(X_1,X_2)}{X_1(1-R_1)} \tilde{\sigma}$.

Computing the PRF in an example

Observe that, although we are able to ensure (under the assumptions made) the existence of the PRF, it is not always possible to obtain the specific expression for such a function. All depends on the possibility of solving (27) to obtain the function ϕ as shown in (29).

In the example presented in section 5, the selected functions F and R enable us to obtain the expressions for ϕ, and hence, for $\tilde{F}$. According to (27) and (28), the variables Z_1 and Z_2 can be defined as

$Z_1 = \frac{X_1 X_2}{X_1+X_2}$, $Z_2 = X_2$. After some simple algebra, we obtain $X_1 = \frac{Z_1 Z_2}{Z_2 - Z_1}$, $X_2 = Z_2$, and substituting these expressions in the production function, we obtain the following expression for the PRF:

$$Y = \tilde{F}(Z_1, Z_2) = F\left(\frac{Z_1 Z_2}{Z_2 - Z_1}, Z_2\right) = \left(\frac{Z_1 Z_2}{Z_2 - Z_1}\right)^{\alpha_1} Z_2^{\alpha_2}.$$

7 Conclusions and further research

This paper has shown that, apart from the usually reported environment-related externalities, there is an important source of externalities linked to recycling, arising from purely technological aspects. One possible market failure comes from the recyclability of some resource itself, and an additional externality comes from the possible technical interactions among resources along the recycling procedure. We have shown how this externalities crucially affect the productive set of the economy and, as a consequence, the optimal use and substitution of natural resources in production, depending on their stock availability, their weight in production and, especially, their natural growth ability.

The results show that the output path follows a new version of the Keynes-Ramsey rule, where the marginal productivity of capital is replaced by the "marginal productivity of natural capital", that results from a weighted sum of the marginal growth of both resources according to their weight in the aggregate technology of the economy. The output level increases (decreases) over time if the marginal productivity of natural capital is larger (smaller) than the discount rate. The speed of this effect depends on the elasticity of temporal substitution. The relative use of resources depends on the difference between the marginal growth of both resources and the velocity of this effect depends on the flexibility of the technology.

If both resources are nonrenewable, they are used in a constant proportion that depends on a measure of relative weight in production and a measure of relative scarcity. If production depends on a renewable and a recyclable resource, the recovery ability of the latter increases its effective availability and permits a more intense use in the short term. In the long term, however, production is more sustainable if it rests more heavily on the renewable resource.

The paper also introduces a generalization of the traditional concept of production function by the joint formalization of production and recycling. As a result, we obtain the Production and Recycling Function (PRF). The PRF shares most of the fundamental properties of a production function and depicts the new production set by capturing the final effect of resources on output as an aggregation of two particular effects: through production and through recycling. This feature of the PRF provides an economically meaningful interpretation for the solution of the production/recycling problem according to traditional concepts from economic theory applied to a context in which conventional production and recycling interact.

Concerning future lines of research, the results obtained in this paper could be extended and enriched by including in the model some further interesting aspects, such as physical capital accumulation, technical change and the waste management aspect of recycling. From an empirical point of view, it would be interesting to estimate the observed effect of the recycling ability on the production possibilities and the effective use of resources.

8 Appendix: Mathematical Results

8.1 Proof of Proposition 1

The *maximized Hamiltonian function* is defined as

$$\mathcal{H}^0(S_1, S_2, \lambda_1, \lambda_2) = \underset{\{X_1, X_2\}}{Max}\, \mathcal{H}(S_1, S_2, X_1, X_2, \lambda_1, \lambda_2).$$

According to the Arrow theorem[14], the necessary Maximum Principle conditions are sufficient for a global maximum for problem (P) if $\mathcal{H}^0$ is concave in (S_1, S_2) for all t, for given λ_1 and λ_2.

The Implicit Function Theorem guarantees that the equation system (5) and (6) can be locally solved for X_1 and X_2 by implicitly defined, $C^{(2)}$ functions $X_1 = \hat{X}_1(\lambda_1, \lambda_2)$, $X_2 = \hat{X}_2(\lambda_1, \lambda_2)$, and the maximized Hamiltonian function can be expressed as

$$\begin{aligned} \mathcal{H}^0 &= U\left[F\left(\hat{X}_1(\lambda_1, \lambda_2), \hat{X}_2(\lambda_1, \lambda_2)\right)\right] \\ &\quad +\lambda_1\left[g_1(S_1) - \hat{X}_1(\lambda_1, \lambda_2) + R\left(\hat{X}_1(\lambda_1, \lambda_2), \hat{X}_2(\lambda_1, \lambda_2)\right)\right] + \lambda_2\left[g_2(S_2) - \hat{X}_2(\lambda_1, \lambda_2)\right]. \end{aligned}$$

For given λ_1 and λ_2, the Hessian matrix of $\mathcal{H}^0$ with respect to S_1, S_2 is

$$Hess = \begin{bmatrix} \lambda_1 g_1'' & 0 \\ 0 & \lambda_2 g_2'' \end{bmatrix}$$

Provided that g_1 and g_2 are concave, we know that $Hess$ is negative semidefinite if and only if $\lambda_1, \lambda_2 \geq 0$. In such a case, $\mathcal{H}^0$ is concave in S_1, S_2 and Arrow sufficient conditions hold.

[14] Arrow and Kurz (1970), proposition 6, p. 45.

8.2 Proof of Proposition 2

Solving (5) for λ_1, substituting in (6) and rearranging, we obtain

$$\lambda_1 = \frac{U' F_1}{1-R_1}, \tag{38}$$

$$\lambda_2 = U' F_2 + R_2 \frac{U' F_1}{1-R_1} = U' \left[\frac{F_2 (1-R_1) + F_1 R_2}{1-R_1} \right]. \tag{39}$$

Deriving (38) and (39) with respect to time and dividing the result by (38) and (39), we have

$$\frac{\dot{\lambda}_1}{\lambda_1} = \frac{1}{U'} \frac{dU'}{dt} + \frac{1}{F_1} \frac{dF_1}{dt} + \frac{1}{(1-R_1)} \frac{dR_1}{dt}, \tag{40}$$

$$\frac{\dot{\lambda}_2}{\lambda_2} = \frac{1}{U'} \frac{dU'}{dt} + \frac{1}{(1-R_1) F_2 + R_2 F_1} \left[-F_2 \frac{dR_1}{dt} + (1-R_1) \frac{dF_2}{dt} + F_1 \frac{dR_2}{dt} + R_2 \frac{dF_1}{dt} \right] + \frac{1}{(1-R_1)} \frac{dR_1}{dt}. \tag{41}$$

Because of the homogeneity assumption on F and R, we know that the functions defined in (10) verify

$$F_1 = f'(x), \qquad F_2 = f(x) - x f'(x), \qquad R_1 = r'(x), \qquad R_2 = r(x) - x r'(x). \tag{42}$$

Deriving (42) with respect to time, we have

$$\frac{dF_1}{dt} = f''(x)\dot{x}, \qquad \frac{dF_2}{dt} = -x \dot{x} f''(x), \qquad \frac{dR_1}{dt} = \dot{x} r''(x), \qquad \frac{dR_2}{dt} = -x \dot{x} r''(x). \tag{43}$$

Using (42) and (43) in (40) and (41), taking (7) into account and rearranging

$$\frac{\dot{\lambda}_1}{\lambda_1} = \frac{1}{U'} \frac{dU'}{dt} + \frac{f'' \dot{x}}{f'} + \frac{r'' \dot{x}}{(1-r')} = \delta - g_1', \tag{44}$$

$$\frac{\dot{\lambda}_2}{\lambda_2} = \frac{1}{U'} \frac{dU'}{dt} + \frac{f''(r-x) - f r''}{f(1-r') + f'(r-x)} \dot{x} + \frac{r''}{(1-r')} \dot{x} = \delta - g_2'. \tag{45}$$

Solving (44) and (45) for $\delta - \frac{r''}{(1-r')}\dot{x} - \frac{1}{U'}\frac{dU'}{dt}$ and equating both results, we have

$$g_1' + \frac{f'' \dot{x}}{f'} = g_2' + \frac{f''(r-x) - f r''}{f(1-r') + f'(r-x)} \dot{x}$$

which, rearranging, becomes $g_2' - g_1' = \dot{x} \frac{f f''(1-r') + f f' r''}{f'[f(1-r') + f'(r-x)]}$ and gives rise to (11).

8.3 Proof of Proposition 3

Adding (44), and (45) and rearranging, we have

$$2 \frac{1}{U'} \frac{dU'}{dt} + \dot{x} J = 2\delta - g_1' - g_2', \tag{46}$$

where $J=\frac{ff'\left(1-r'\right)r''+2r''\left(f'\right)^2(r-x)+ff''\left(1-r'\right)^2+2f'f''\left(1-r'\right)(r-x)}{f'(1-r')[f(1-r')+f'(r-x)]}$. Using (11) in (46) we obtain $2\frac{1}{U'}\frac{dU'}{dt}+x\hat{\sigma}J\left[g_1'-g_2'\right]=2\delta-g_1'-g_2'$, and solving for $\frac{1}{U'}\frac{dU'}{dt}$ we have

$$\frac{1}{U'}\frac{dU'}{dt}=\frac{1}{2}\left[2\delta-g_1'\left(1+x\hat{\sigma}J\right)-g_2'\left(1-x\hat{\sigma}J\right)\right], \tag{47}$$

where, taking (12) into account, we know that

$$x\hat{\sigma}J=\frac{-\left\{ff'\left(1-r'\right)r''+2r''\left(f'\right)^2(r-x)+ff''\left(1-r'\right)^2+2f'f''\left(1-r'\right)(r-x)\right\}}{\left(1-r'\right)\left[ff''\left(1-r'\right)+ff'r''\right]}$$

and, substituting in (47) and rearranging, we obtain

$$\frac{1}{U'}\frac{dU'}{dt}=\left[\delta-\left(\hat{\xi}_1g_1'+\hat{\xi}_2g_2'\right)\right], \tag{48}$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are defined in (14) . Taking $\frac{dU'}{dt}=U''\dot{Y}$ into account and using the definition of $\eta\left(Y\right)$, (48) becomes (13). Adding up the expressions for $\hat{\xi}_1$ and $\hat{\xi}_2$ in (14) we obtain $\hat{\xi}_1+\hat{\xi}_2=1$

8.4 Proof of Proposition 4

If both resources are nonrenewable, (7) becomes $\dot{\lambda}_i=\delta\lambda_i$, which is solved by $\lambda_i=\lambda_i\left(0\right)e^{\delta t}$, $\lambda_i\left(0\right)$ being the value of λ_i at $t=0$. Dividing the expressions for λ_2 and λ_1 we have $\Lambda=\frac{\lambda_2(0)}{\lambda_1(0)}$, which is constant. Substituting (44) and (45) in the definition of Λ, we obtain $\Lambda=\frac{F_2(1-R_1)+F_1R_2}{F_1}$ which, using (10) and (42) and rearranging, can be expressed as $\Lambda=\frac{f\left(1-r'\right)+f'r}{f'}-x$ or, using (10) and (42) again, jointly with the definition of x, $x=x\frac{F(1-R_1)+F_1R}{X_1F_1}-\Lambda$. Using the Euler result for homogeneous functions, $F\left(X_1,X_2\right)=X_1F_1\left(X_1,X_2\right)+X_2F_2\left(X_1,X_2\right)$, and solving for x, we obtain (15). Furthermore, given that x and Λ are constant, ψ is constant as well

8.5 Example with two nonrenewable resources and recycling

The problem to solve is

$$\begin{aligned}
&\underset{\{X_1,X_2\}}{Max}\int_0^{\infty}\tfrac{1}{1-\eta}\left(X_1^{\alpha_1(1-\eta)}X_2^{\alpha_2(1-\eta)}\right)e^{-\delta t}dt\\
&s.a.:\\
&\qquad\dot{S}_1=-X_1+\frac{X_1^2}{X_1+X_2}=-\frac{X_1X_2}{X_1+X_2},\\
&\qquad\dot{S}_2=-X_2,\\
&\qquad S_i(0)=S_i^0, \qquad\qquad\qquad i=1,2\\
&\qquad 0\leq X_i\leq S_i, \qquad\qquad\qquad i=1,2
\end{aligned}$$

Given that both resources are essential for production, we can discard as optimal any corner solution with $X_1 = 0$ or $X_2 = 0$. The current-value Hamiltonian takes the expression

$$\mathcal{H} = \frac{1}{1-\eta}\left(X_1^{\alpha_1(1-\eta)} X_2^{\alpha_2(1-\eta)}\right) - \lambda_1 \frac{X_1 X_2}{X_1 + X_2} - \lambda_2 X_2.$$

The shadow price of a nonrenewable resource grows at a constant rate equal to δ, so that $\lambda_i = \lambda_i(0)\, e^{\delta t}$. The first order conditions for the maximization of the Hamiltonian with respect to X_1 and X_2 are

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial X_1} &= \alpha_1 X_1^{\alpha_1(1-\eta)-1} X_2^{\alpha_2(1-\eta)} - \lambda_1 \frac{X_2^2}{(X_1+X_2)^2} = 0, \\
\frac{\partial \mathcal{H}}{\partial X_2} &= \alpha_2 X_1^{\alpha_1(1-\eta)} X_2^{\alpha_2(1-\eta)-1} - \lambda_1 \frac{X_1^2}{(X_1+X_2)^2} - \lambda_2 = 0
\end{aligned}$$

and rearranging we obtain

$$\lambda_1 = \alpha_1 X_1^{\alpha_1(1-\eta)-1} X_2^{\alpha_2(1-\eta)-2} (X_1+X_2)^2, \tag{49}$$

$$\lambda_2 = \alpha_2 X_1^{\alpha_1(1-\eta)} X_2^{\alpha_2(1-\eta)-1} - \alpha_1 X_1^{\alpha_1(1-\eta)+1} X_2^{\alpha_2(1-\eta)-2}, \tag{50}$$

which, evaluated at $t = 0$, give

$$\begin{aligned}
\lambda_1(0) &= \alpha_1 X_1(0)^{\alpha_1(1-\eta)-1} X_2(0)^{\alpha_2(1-\eta)-2} (X_1(0) + X_2(0))^2, \qquad \text{and} \\
\lambda_2(0) &= \alpha_2 X_1(0)^{\alpha_1(1-\eta)} X_2(0)^{\alpha_2(1-\eta)-1} - \alpha_1 X_1(0)^{\alpha_1(1-\eta)+1} X_2(0)^{\alpha_2(1-\eta)-2}.
\end{aligned}$$

From the results for two nonrenewable resources shown in section 4, we know that $\frac{\dot{Y}}{Y} = \frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2} = \frac{-\delta}{\eta}$. The general solutions for these differential equations are $Y = Y(0)\, e^{-\frac{\delta}{\eta}t}$, $X_1 = X_1(0)\, e^{-\frac{\delta}{\eta}t}$ and $X_2 = X_2(0)\, e^{-\frac{\delta}{\eta}t}$. From the transversality conditions for problem (P), we know that both resources become exhausted under the optimal solution and essentiality imply that their exhaustion must be simultaneous. Let T denote the exhaustion time. The optimal value of T comes from the transversality condition $\mathcal{H}(T) = 0$. Evaluating the solutions for X_1, X_2, λ_1 and λ_2 at instant T and substituting in the Hamiltonian, we conclude that $T = \infty$ and both resources exhaust asymptotically.

Substituting the solution for X_2 in the state equation for resource 2 and using the initial condition $S_2(0) = S_2^0$, we obtain $S_2 = S_2^0 + \frac{\eta}{\delta} X_2(0)\left[e^{-\frac{\delta}{\eta}t} - 1\right]$ and, using the final condition $\lim_{t\to\infty} S_2(t) = 0$, we have the initial value for X_2, $X_2(0) = \frac{\delta}{\eta} S_2^0$. Substituting the solutions for X_1 and X_2 in the state equation for resource 1, we have $\dot{S}_1 = -\frac{X_1(0)X_2(0)}{X_1(0)+X_2(0)} e^{-\frac{\delta}{\eta}t}$ that, using the initial condition $S_1(0) = S_1^0$, is solved by $S_1 = S_1^0 + \frac{\eta}{\delta} \frac{X_1(0)X_2(0)}{X_1(0)+X_2(0)}\left[e^{-\frac{\delta}{\eta}t} - 1\right]$ and, using the final condition $\lim_{t\to\infty} S_1(t) = 0$, the expression for $X_2(0)$ and rearranging, provides $X_1(0) = \frac{\delta}{\eta} \frac{S_1^0 S_2^0}{S_2^0 - S_1^0}$. Deriving the solutions for X_1 and Y with respect to S_1^0 and S_2^0, we obtain the following sensitivity analysis results:

$$
\begin{aligned}
\frac{\partial X_1}{\partial S_1^0} &= \frac{\delta}{\eta}\frac{\left(S_2^0\right)^2}{\left(S_2^0-S_1^0\right)^2}e^{-\frac{\delta}{\eta}t} > 0, \qquad \frac{\partial X_1}{\partial S_2^0} = -\frac{\delta}{\eta}\frac{\left(S_1^0\right)^2}{\left(S_2^0-S_1^0\right)^2}e^{-\frac{\delta}{\eta}t} < 0,\\
\frac{\partial Y}{\partial S_1^0} &= \alpha_1\frac{\delta}{\eta}\left(S_2^0\right)^2\left(S_1^0\right)^{\alpha_1-1}\left(\frac{1}{S_2^0-S_1^0}\right)^{\alpha_1+1} e^{-\frac{\delta}{\eta}t} > 0,\\
\frac{\partial Y}{\partial S_2^0} &= \frac{\delta}{\eta}\left[\left(\frac{S_1^0}{S_2^0-S_1^0}\right)^{\alpha_1} - S_2^0\alpha_1\left(\frac{S_1^0}{S_2^0-S_1^0}\right)^{\alpha_1-1}\frac{S_1^0}{\left(S_2^0-S_1^0\right)^2}\right]e^{-\frac{\delta}{\eta}t} =\\
&= \frac{\delta}{\eta}\left(\frac{S_1^0}{S_2^0-S_1^0}\right)^{\alpha_1}\frac{S_2^0\left(1-\alpha_1\right)-S_1^0}{S_2^0-S_1^0}e^{-\frac{\delta}{\eta}t} \geq 0 \Leftrightarrow S_2^0 \geq \frac{S_1^0}{\left(1-\alpha_1\right)}.
\end{aligned}
$$

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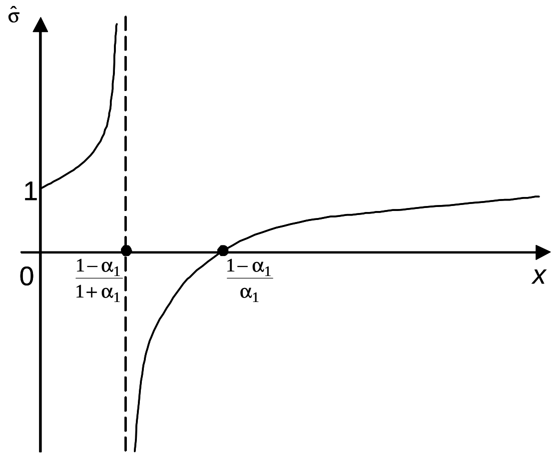

Figure 1. Shape of $\hat{\sigma}$ as a function of x.

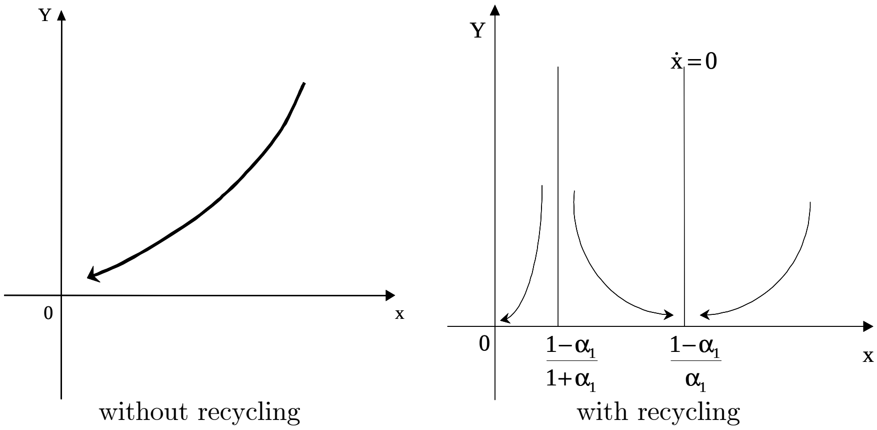

Figure 2. Phase diagrams with renewable and nonrenewable resource

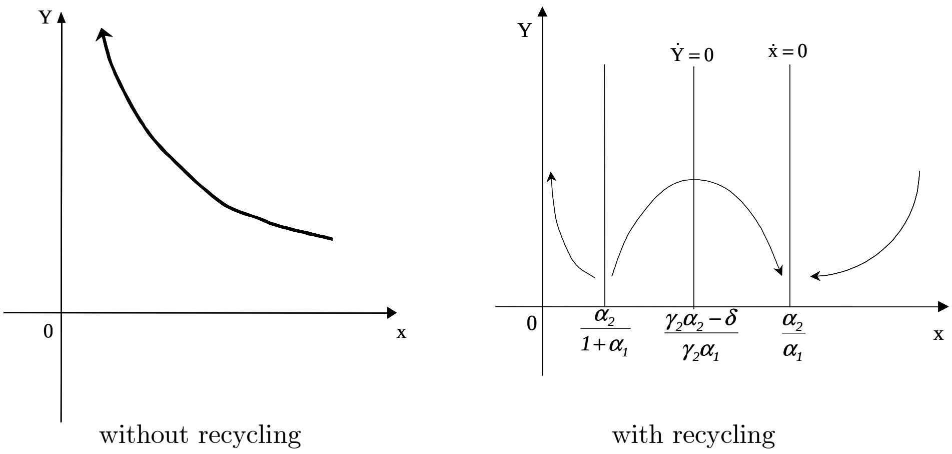

Figure 3. Phase diagrams with renewable and nonrenewable resource

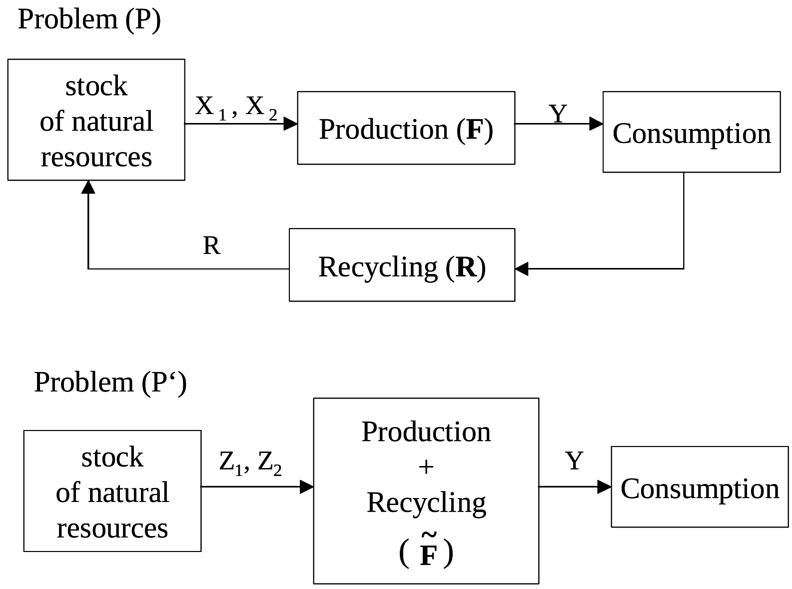

Figure 4: Comparing (P) and (P’)